Applications and generalizations of Fisher-Hartwig asymptotics

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Fisher-Hartwig asymptotics refers to the large n form of a class of Toeplitz determinants with singular generating functions. This class of Toeplitz determinants occurs in the study of the spin-spin correlations for the two-dimensional Ising model, and the ground state density matrix of the impenetrable Bose gas, amongst other problems in mathematical physics. We give a new application of the original Fisher-Hartwig formula to the asymptotic decay of the Ising correlations above T_c, while the study of the Bose gas density matrix leads us to generalize the Fisher-Hartwig formula to the asymptotic form of random matrix averages over the classical groups and the Gaussian and Laguerre unitary matrix ensembles. Another viewpoint of our generalizations is that they extend to Hankel determinants the Fisher-Hartwig asymptotic form known for Toeplitz determinants.

Encomium

In celebration of Freeman Dyson on his

$$\Big(32(\tfrac{40960001}{25600})^3 - 6(\tfrac{40960001}{25600}) + \sqrt{[32(\tfrac{40960001}{25600})^3 - 6(\tfrac{40960001}{25600})]^2 - 1}\Big)^{1/6} \text{th}$$

birthday[1].

Dyson's legendary works [2] on random matrices are now standards in physics, mathematics and fields far(ther) afield. These works and, in particular his powerful log-Coulomb gas model, developed to liberate the mathematics where none yet exists, are the very essential tools in our ongoing pursuits [3].

1 Introduction

Fisher-Hartwig asymptotics refers to the large n form of a class of Toeplitz determinants $D_n[g]$. By definition, the entries of the latter depend only on the difference of the row and column indices, and thus

$$D_n[g] = \det[g_{j-k}]_{j,k=1,\dots,n} \tag{1.1}$$

for some $\{g_k\}_{k=0,\pm1,\pm2,\dots}$. Crucial to the structure of the asymptotic form of (1.1) are analytic properties of the so called symbol

$$g(\theta) := \sum_{n=-\infty}^{\infty} g_n e^{in\theta}, \tag{1.2}$$

[1]This radical appears in the works on Ramanujan [1], who is very dear to Dyson.

or more particularly the decay of the Fourier coefficients of $\log g(\theta)$. Explicitly, let

$$\log g(\theta) = \sum_{p=-\infty}^{\infty} c_p e^{ip\theta}. \tag{1.3}$$

Then if

$$\sum_{p=-\infty}^{\infty} |p| c_p c_{-p} < \infty \tag{1.4}$$

a strong form of the Szegö limit theorem (see e.g. [4, 5]) asserts that for $n \to \infty$

$$D_n[g] = \exp\Big(nc_0 + \sum_{k=1}^{\infty} k c_k c_{-k} + \mathrm{o}(1)\Big). \tag{1.5}$$

Two cases for which (1.4) will not hold are when $g(\theta)$ has a jump discontinuity or a zero for some $-\pi < \theta \le \pi$. It is for such singular symbols (in the case of a zero it is the logarithm of the symbol which is singular) that Fisher and Hartwig [6] sought the asymptotic form of (1.1). Symbols with singularities of this type have the functional form

$$\begin{aligned} \log g(\theta) &= \log a(\theta) - i \sum_{r=1}^{R} b_r \arg e^{i(\theta_r + \pi - \theta)} + \sum_{r=1}^{R} a_r \log |2 - 2\cos(\theta - \theta_r)| \\ &= \log a(\theta) + \sum_{r=1}^{R} \Big((a_r + b_r) \log(1 + e^{i(\theta - (\theta_r + \pi))}) + (a_r - b_r) \log(1 + e^{i(\theta_r + \pi - \theta)}) \Big). \end{aligned} \tag{1.6}$$

Here $-\pi < \arg z \le \pi$ and $a(\theta)$ is assumed to be sufficiently smooth that if we write

$$\log a(\theta) = \sum_{p=-\infty}^{\infty} c_p e^{ip\theta} \tag{1.7}$$

(cf. (1.3)) then the condition (1.4) holds. By using data following from the fact that special cases of (1.6) correspond to Toeplitz determinant expressions for the spin-spin correlation in the two-dimensional Ising model at criticality (see Section 2 below), the asymptotic form of which had previously been calculated [7], Fisher and Hartwig [6] conjectured that for some range of parameter values $\{a_r\}_{r=1,\dots,R}$, $\{b_r\}_{r=1,\dots,R}$,

$$D_n[g] \underset{n \to \infty}{\sim} e^{c_0 n} e^{\sum_{r=1}^{R} (a_r^2 - b_r^2) \log n} E \tag{1.8}$$

where E is independent of n. Subsequently this was proved for various ranges of parameter values (see e.g. [8]) and furthermore the constant was determined to be given by

$$\begin{aligned} E &= e^{\sum_{k=1}^{\infty} k c_k c_{-k}} \prod_{r=1}^{R} e^{-(a_r + b_r) \log a_-(\theta_r)} e^{-(a_r - b_r) \log a_+(\theta_r)} \\ &\quad \times \prod_{1 \le r \ne s \le R} (1 - e^{i(\theta_s - \theta_r)})^{-(a_r + b_r)(a_s - b_s)} \prod_{r=1}^{R} \frac{G(1 + a_r + b_r) G(1 + a_r - b_r)}{G(1 + 2a_r)} \end{aligned} \tag{1.9}$$

where G is the Barnes G-function and

$$\log a_+(\theta) := \sum_{p=1}^{\infty} c_p e^{ip\theta}, \qquad \log a_-(\theta) := \sum_{p=-\infty}^{-1} c_p e^{ip\theta}. \tag{1.10}$$

Our interest is in applications and generalizations of the Fisher-Hartwig asymptotic formula (1.8). We begin in Section 2 with an application of (1.8) to the calculation of the asymptotic form of the spin-spin

correlation for the two-dimensional Ising model above criticality. In Section 3 the well known equivalence of the Toeplitz determinant (1.1) to a random matrix average over the unitary group $U(n)$ is revised. This average is in turn equivalent to the partition function of the one-component log-gas on a circle, subject to a one-body potential with Boltzmann factor $g(\theta)$ at the special coupling $\beta = 2$. As such there is a natural generalization for couplings $\beta > 0$, and in the case $b_r = 0$, $r = 1, \ldots, R$ this can be used to predict the corresponding generalization of (1.8). Moreover, in the special case $a(\theta) = 1$, $R = 1$ the sought asymptotic form can be deduced from an exact formula valid for general a_r, b_r. This can be used to extend the conjectured generalization of (1.8) to non-zero b_r.

In Section 4 we recall the problem of computing the asymptotic form of the density matrix for impenetrable bosons in Dirichlet and Neumann boundary conditions. This is immediately identifiable as an average over the classical groups $Sp(N)$ and $O^+(2N)$ respectively, with the function being averaged over having two zeros, and thus analogous to the random matrix formulation of the Toeplitz determinant (1.1) with symbol (1.6) in the case $R = 2$, $b_r = 0$. We point out that the same class of averages over the groups $O^+(2N + 1)$ or $O^-(2N + 1)$ result from considering the density matrix for the impenetrable Bose gas in the case of mixed Dirichlet and Neumann boundary conditions. In [9] the sought asymptotics were calculated on the basis of a combination of analytic and log-gas arguments, and a Fisher-Hartwig type generalization (with $b_r = 0$) conjectured. The conjecture of [9] can used to predict the asymptotic form in the case of mixed Dirichlet and Neumann boundary conditions. Moreover we show that this asymptotic form can be proved by making use of asymptotic formulas recently obtained [10] for Toeplitz + Hankel determinants

$$\det[a_{j-k} + a_{j+k+1}]_{j,k=0,\ldots,n-1} \tag{1.11}$$

in the case of singular generating functions (1.6).

In addition to averages over the classical groups, the study of the density matrix for impenetrable bosons naturally leads to the question of obtaining the asymptotic form of averages over the eigenvalue probability density function for the GUE and LUE, in the case that the function being averaged over has zeros. Here the GUE denotes the Gaussian unitary ensemble of random Hermitian matrices, and the LUE denotes the Laguerre unitary ensemble of positive definite matrices with complex entries. These random matrix averages are equivalent to pure Hankel determinants

$$\det[a_{j+k}]_{j,k=0,\ldots,n-1}, \qquad a_n = \int_{-\infty}^{\infty} a(x) x^n \, d\mu(x) \tag{1.12}$$

where $d\mu(x) = e^{-x^2} dx$ for the GUE and $d\mu(x) = x^a e^{-x} dx$, $x > 0$ for the LUE. Conjectures for such asymptotic forms are given in Section 5. The paper ends with some concluding remarks on the universal form for Hankel asymptotics in Section 6, and attention is also drawn to the fluctuation formula perspective of our asymptotic results.

2 Spin-spin correlations for the two-dimensional Ising model

In the two-dimensional Ising model on a square lattice each site (i, j) of the lattice exists in one of two possible states $\sigma_{ij} = \pm 1$ with coupling between nearest neighbours in the horizontal and vertical directions. Explicitly, the joint probability density function for a particular configuration $\{\sigma_{ij}\}$ of the states on a $(2N + 1) \times (2N + 1)$ lattice is given by

$$P_{2N+1}(\{\sigma_{ij}\}) = \frac{1}{Z_{2N+1}} \exp\Big(K_1 \sum_{j=-N}^{N} \sum_{i=-N}^{N-1} \sigma_{ij}\sigma_{i+1\,j} + K_2 \sum_{i=-N}^{N} \sum_{j=-N}^{N-1} \sigma_{ij}\sigma_{i\,j+1} \Big) \tag{2.1}$$

where Z_{2N+1} is the normalization. The spin-spin correlation function between the spin σ_{00} at the centre of the lattice, and the spin $\sigma_{i^* j^*}$ at site (i^*, j^*) is, in the infinite lattice limit, defined as

$$\langle \sigma_{00}\sigma_{i^*j^*}\rangle = \lim_{N\to\infty} \sum_{\{\sigma_{ij}\}} \sigma_{00}\sigma_{i^*j^*} P_{2N+1}(\{\sigma_{ij}\}). \tag{2.2}$$

Onsager knew of, but never published (see instead e.g. [11]) a Toeplitz determinant form for the case of (2.2) for which $(i^*, j^*) = (n, n)$ and thus lies on the diagonal. Explicitly

$$\langle \sigma_{00}\sigma_{nn}\rangle = \det[a_{i-j}]_{i,j=1,\dots,n}, \qquad a_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)e^{-ip\theta}\, d\theta \tag{2.3}$$

where

$$h(\theta) := \Big(\frac{1+(1/k)e^{-i\theta}}{1+(1/k)e^{i\theta}}\Big)^{1/2}, \qquad k = \sinh 2K_1 \sinh 2K_2. \tag{2.4}$$

Also, in the case of (2.2) with $(i^*, j^*) = (0, n)$ so that the two spins lie in the same row, Onsager and Kaufmann [12] expressed (2.2) as the sum of two Toeplitz determinants. A different approach to this problem was undertaken by Potts and Ward [13], who obtained instead the single Toeplitz determinant form

$$\langle \sigma_{00}\sigma_{0n}\rangle = \det[\tilde{a}_{i-j}]_{i,j=1,\dots,n}, \qquad \tilde{a}_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{h}(\theta)e^{-ip\theta}\, d\theta \tag{2.5}$$

where

$$\tilde{h}(\theta) := \Big(\frac{(1+\alpha_1 e^{i\theta})(1+\alpha_2 e^{-i\theta})}{(1+\alpha_1 e^{-i\theta})(1+\alpha_2 e^{i\theta})}\Big)^{1/2} \tag{2.6}$$

with

$$\alpha_1 := e^{-2K_2}\tanh K_1, \qquad \alpha_2 := \frac{e^{-2K_2}}{\tanh K_1}.$$

The formula obtain in [12] was shown to be identical to (2.3), (2.4) by Montroll, Potts and Ward [14]. We remark that if α_1, α_2 in (2.6) and k in (2.4) are regarded as parameters not specified by K_1, K_2, then setting $\alpha_1 = 0, \alpha_2 = 1/k$ in the former gives (2.4).

A detailed study of the asymptotic form of (2.6) was undertaken by Wu [7]. Indeed, it was the asymptotic form of (2.5) at the critical coupling

$$\alpha_1 < \alpha_2 = 1 \tag{2.7}$$

obtained in [7] which, partially at least, inspired the formulation of the Fisher-Hartwig asymptotic formula (1.8) [6]. To see how (1.8) relates to (2.5) with parameters (2.7), note

$$\log \tilde{h}(\theta)\Big|_{\alpha_2=1} = \log\Big(\frac{1+\alpha_1 e^{i\theta}}{1+\alpha_1 e^{-i\theta}}\Big)^{1/2} + i\arg e^{-i\theta/2}. \tag{2.8}$$

For $|\alpha_1| < 1$ this has the structure of (1.6) with

$$a(\theta) = \Big(\frac{1+\alpha_1 e^{i\theta}}{1+\alpha_1 e^{-i\theta}}\Big)^{1/2}, \;\; R = 1, \;\; b_r = -\frac{1}{2}, \;\; a_r = 0, \;\; \theta_r = -\pi.$$

Recalling the definitions (1.3) (with $g(\theta)$ replaced by $a(\theta)$) and (1.10), application of (1.8) implies

$$\langle \sigma_{00}\sigma_{0n}\rangle\Big|_{\substack{\alpha_2=1\\ \alpha_1<1}} \underset{n\to\infty}{\sim} \Big(\frac{1+\alpha_1}{1-\alpha_1}\Big)^{1/4} \frac{\sqrt{\pi}G^2(1/2)}{n^{1/4}} \tag{2.9}$$

where use has been made of the functional equation

$$G(z+1) = \Gamma(z)G(z),$$

in agreement with the result of Wu.

The high temperature phase corresponds to couplings

$$\alpha_1 < 1 < \alpha_2, \qquad \alpha_1\alpha_2 < 1. \tag{2.10}$$

In this case $\log \tilde{h}(\theta)$ is of the form (1.6) with

$$a(\theta) = \Big(\frac{(1+\alpha_1 e^{i\theta})(1+e^{i\theta}/\alpha_2)}{(1+\alpha_1 e^{-i\theta})(1+e^{-i\theta}/\alpha_2)} \Big)^{1/2}, \;\; R=1, \;\; b_r=-1, \;\; a_r=0, \;\; \theta_r=-\pi. \tag{2.11}$$

With $R=1, b_r=-1, a_r=0$ we see that the Fisher-Hartwig asymptotic formula (1.8) breaks down because according to (1.9) the constant E contains the factor $G(0)=0$ and thus vanishes. To obtain the asymptotics in this case the approach taken in [7] was to relate it back to the original strong Szegö theorem, multiplied by an auxilarly factor. Here we will show that by transforming (2.6), a form of $\log \tilde{h}(\theta)$ can be obtained which has the general structure (1.6) but is distinct from the specification (2.11). We will see that applying the Fisher-Hartwig formula then correctly reproduces the result of Wu for the leading asymptotic decay in the high temperature phase.

For this purpose, let us introduce the notation $f(\theta) \equiv g(\theta)$ to mean that

$$\int_{-\pi}^{\pi} f(\theta) e^{-ip\theta}\, d\theta = c^{-p} \int_{-\pi}^{\pi} g(\theta) e^{-ip\theta}\, d\theta$$

for some c independent of p. According to the definitions (2.6) and (2.11) we have $\tilde{h}(\theta) = e^{-i\theta}a(\theta)$. Now, since with $z=e^{i\theta}$, $\tilde{h}(\theta)$ is an analytic function of z in the annulus $1/\alpha_2 < |z| < \alpha_2$, by Cauchy's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{h}(\theta) e^{-ip\theta}\, d\theta = \int_{\mathcal{C}} \tilde{h}(\theta) z^{-p} \frac{dz}{2\pi i z}$$

for any simple closed contour encircling the origin in this annulus. Choosing $\mathcal{C}$ to be the circle with radius α_2 (the outer boundary of the annulus) shows

$$\begin{aligned} \tilde{h}(\theta) &\equiv \frac{e^{-i\theta}}{\alpha_2}\Big(\frac{1+\alpha_1\alpha_2 e^{i\theta}}{1+(\alpha_1/\alpha_2)e^{-i\theta}} \Big)^{1/2} \frac{(1+e^{i\theta})^{1/2}}{(1+e^{-i\theta}/\alpha_2^2)^{1/2}} \\ &= \frac{1}{\alpha_2}\frac{1}{(1+e^{-i\theta}/\alpha_2^2)^{1/2}} \Big(\frac{1+\alpha_1\alpha_2 e^{i\theta}}{1+(\alpha_1/\alpha_2)e^{-i\theta}} \Big)^{1/2} e^{-3i\theta/4}|1+e^{i\theta}|^{1/2}. \end{aligned}$$

This is of the form (1.6) with

$$a(\theta) = \frac{1}{\alpha_2}\frac{1}{(1+e^{-i\theta}/\alpha_2^2)^{1/2}} \Big(\frac{1+\alpha_1\alpha_2 e^{i\theta}}{1+(\alpha_1/\alpha_2)e^{-i\theta}} \Big)^{1/2}, \;\; R=1, \;\; b_r=-\frac{3}{4}, \;\; a_r=\frac{1}{4}, \;\; \theta_r=-\pi. \tag{2.12}$$

Application of (1.8) implies

$$\langle \sigma_{00}\sigma_{0n} \rangle \Big|_{\substack{\alpha_1<1<\alpha_2 \\ \alpha_1\alpha_2<1}} \underset{n\to\infty}{\sim} \frac{\alpha_2^{-n}}{(\pi n)^{1/2}} (1-\alpha_1^2)^{1/4}(1-\alpha_2^{-2})^{-1/4}(1-\alpha_1\alpha_2)^{-1/2} \tag{2.13}$$

in agreement with the result of Wu [7]. Moreover the Fisher-Hartwig formula (1.8) with $R=1$ has been proved [15] for parameter values satisfying all three of the inequalities

$$\mathrm{Re}\, a_1 \ge 0, \qquad \mathrm{Re}\, a_1 + \mathrm{Re}\, b_1 > -1, \qquad \mathrm{Re}\, a_1 - \mathrm{Re}\, b_1 > -1.$$

These inequalities are satisfied by the parameters in (2.12) and so the Fisher-Hartwig formula provides a proof of (2.13).

3 β-generalization of the Fisher-Hartwig formula

It is well known, and easy to verify, that the Toeplitz determinant (1.2) can be written as a random matrix average according to

$$D_n[g] = \Big\langle \prod_{l=1}^{n} g(\theta_l) \Big\rangle_{U(n)}. \tag{3.1}$$

Here $U(n)$ refers to the eigenvalue probability density function for the unitary group

$$\frac{1}{(2\pi)^n n!} \prod_{1 \le j < k \le n} |e^{i\theta_k} - e^{i\theta_j}|^2, \qquad -\pi < \theta_l \le \pi. \tag{3.2}$$

As first noted by Dyson [2], (3.2) is proportional to the Boltzmann factor for the one-component log-potential Coulomb gas on a circle, at the special coupling $\beta = 2$. From the log-gas viewpoint a natural generalization of (3.2) is the probability density function CβE$_n$ proportional to the Boltzmann factor for the same statistical mechanical system but with general coupling $\beta > 0$,

$$\frac{1}{(2\pi)^n C_{n,\beta}} \prod_{1 \le j < k \le n} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \qquad C_{n,\beta} = \frac{\Gamma(n\beta/2+1)}{(\Gamma(\beta/2+1))^n}. \tag{3.3}$$

The identity (3.1) then allows us to formulate a β-generalization of the Toeplitz determinant (1.2) as the average

$$D_n^{(\beta)}[g] := \Big\langle \prod_{l=1}^{n} g(\theta_l) \Big\rangle_{\mathrm{C}\beta\mathrm{E}_n}. \tag{3.4}$$

Choosing $g(\theta)$ according to (1.6) in this we obtain a natural β-generalization of the Toeplitz determinant with a Fisher-Hartwig symbol. In the case $b_r = 0$, $r = 1, \dots, R$, the log-gas viewpoint can be used to conjecture the corresponding analogue of the asymptotic formula (1.8).

Let

$$Z_n^{(\beta)}[g(\theta)] := \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_n \prod_{l=1}^{n} g(\theta_l) \prod_{1 \le j < k \le n} |e^{i\theta_k} - e^{i\theta_j}|^\beta. \tag{3.5}$$

As the first step, guided by both the log-gas viewpoint and the structure of (1.8), we conjecture the factorization

$$\frac{Z_n^{(\beta)}[a(\theta)\prod_{j=1}^R |e^{i\theta} - e^{i\phi_j}|^{q_j\beta}]}{Z^{(\beta)}_{n+\sum_{j=1}^R q_j}[1]} \sim e^{-\sum_{j=1}^R q_j \log a(\theta_j)} \frac{Z_n^{(\beta)}[a(\theta)]}{Z^{(\beta)}_{n+\sum_{j=1}^R q_j}[1]} \frac{Z_n^{(\beta)}[\prod_{j=1}^R |e^{i\theta} - e^{i\phi_j}|^{q_j\beta}]}{Z^{(\beta)}_{n+\sum_{j=1}^R q_j}[1]} \tag{3.6}$$

From the work of Johansson [5, 16], with the Fourier expansion of $\log a(\theta)$ specified by (1.7) and assuming the coefficients satisfy (1.4), it has been proved for general $\beta > 0$ that

$$\frac{Z_{n+Q}^{(\beta)}[a(\theta)]}{Z_{n+Q}^{(\beta)}[1]} \sim e^{c_0(n+Q)} e^{(2/\beta)\sum_{k=1}^\infty k c_k c_{-k}}. \tag{3.7}$$

Regarding the second ratio on the right hand side of (3.6), as first noted in [17, 18] and revised in [20], the log-gas viewpoint suggests that for $n \to \infty$ we have the factorization

$$\prod_{1 \le j < k \le R} |e^{i\theta_k} - e^{i\theta_j}|^{\beta q_j q_k} \frac{Z_n^{(\beta)}[\prod_{j=1}^R |e^{i\theta} - e^{i\phi_j}|^{q_j\beta}]}{Z^{(\beta)}_{n+\sum_{j=1}^R q_j}[1]} \sim \prod_{j=1}^R \frac{Z_n^{(\beta)}[\prod_{j=1}^R |e^{i\theta} - e^{i\phi_j}|^{q_j\beta}]}{Z^{(\beta)}_{n+q_j}[1]} \tag{3.8}$$

The large-n expansion of a ratio closely related to the product on the right hand side of (3.8) is known for β rational, in particular

$$\beta/2 = s/r, \qquad s \text{ and } r \text{ relatively prime}. \tag{3.9}$$

Thus we have [18]

$$\prod_{j=1}^{R}\frac{Z_n^{(\beta)}[\prod_{j=1}^{R}|e^{i\theta}-e^{i\phi_j}|^{q_j\beta}]}{Z_n^{(\beta)}[1]}\sim n^{q^2\beta/2}A_q, \tag{3.10}$$

where

$$A_q:=r^{-q^2\beta/2}\prod_{\nu=0}^{r-1}\prod_{p=0}^{s-1}\frac{G^2(q/r+\nu/r-p/s+1)}{G(2q/r+\nu/r-p/s+1)G(\nu/r-p/s+1)}. \tag{3.11}$$

Finally, the formula for $C_{n,\beta}$ in (3.3) together with Stirling's formula shows

$$\frac{Z_n^{(\beta)}[1]}{Z_{n+q}^{(\beta)}[1]}\sim(\Gamma(\beta/2+1))^q(n\beta/2)^{-q\beta/2}. \tag{3.12}$$

Combining the above results gives the sought β-generalization of the Fisher-Hartwig formula in the case $b_r=0$.

Conjecture 1. *Let β be rational and of the form (3.9), and let $a(\theta)$ be as assumed for the validity of (3.7). For $q_j\beta>-1$ we expect*

$$\Big\langle\prod_{l=1}^{N}\Big(a(\theta_l)\prod_{j=1}^{R}|e^{i\theta_l}-e^{i\phi_j}|^{q_j\beta}\Big)\Big\rangle_{\mathrm{C}\beta\mathrm{E}_n}\underset{n\to\infty}{\sim}e^{c_0(n+\sum_{j=1}^{R}q_j)}n^{(\beta/2)\sum_{j=1}^{R}q_j^2}E^{(\beta)} \tag{3.13}$$

where, with A_q specified by (3.11),

$$E^{(\beta)}=e^{-\sum_{j=1}^{R}q_j\log a(\theta_j)}e^{(2/\beta)\sum_{k=1}^{\infty}kc_kc_{-k}}\prod_{1\le j<k\le R}|e^{i\theta_k}-e^{i\theta_j}|^{-\beta q_jq_k}\prod_{j=1}^{R}A_{q_j}. \tag{3.14}$$

It is of interest to extend Conjecture 1 to include a factor

$$\prod_{j=1}^{R}e^{-i(\beta/2)b_r\arg e^{i(\phi_j+\pi-\theta)}} \tag{3.15}$$

in the average, and so obtain a β-generalization of the Fisher-Hartwig formula for general parameters. Although we don't have a log-gas interpretation of the factor (3.15), the case $R=1$ substituted in (3.13) with $a(\theta)=$ gives an average which can be evaluated in closed form, and the corresponding asymptotics computed for rational β. This together with the structure of the original Fisher-Hartwig formula (1.8), (1.9) allows us to formulate the sought β-generalization.

Now, by rotational invariance, independent of the value of ϕ

$$\begin{aligned}\Big\langle\prod_{l=1}^{n}e^{-i(\beta/2)b\arg e^{i(\phi+\pi-\theta_l)}}|e^{i\theta_l}-e^{i\phi}|^{\beta q}\Big\rangle_{\mathrm{C}\beta\mathrm{E}_n}&=\Big\langle\prod_{l=1}^{n}e^{i\beta b\theta_l/2}|1+e^{i\theta_l}|^{\beta q}\\ \Big\rangle_{\mathrm{C}\beta\mathrm{E}_n}&=\frac{Z_n^{(\beta)}[e^{i\beta b\theta/2}|1+e^{i\theta}|^{\beta q}]}{Z_n^{(\beta)}[1]}.\end{aligned} \tag{3.16}$$

But, from the theory of the Selberg integral (see e.g. [3]), we know the right hand side of (3.16) has the explicit gamma function evaluation

$$\frac{f_n(2cq,c)}{f_n(c(q+b),c)f_n(c(q-b),c)}\qquad\text{where}\quad f_n(\alpha,c):=\prod_{j=0}^{n-1}\frac{(\alpha+jc)!}{(jc)!},\quad c:=\beta/2. \tag{3.17}$$

For $c \in \mathbb{Z}^+$ it was shown in [18] that

$$f_n(\alpha, c) \underset{n\to\infty}{\sim} \exp(\alpha n \log n) c^{\alpha n} e^{-\alpha n} n^{-(c-1)\alpha/2+\alpha^2/2c} \prod_{p=0}^{c-1} \frac{G(-p/c+1)}{G((\alpha-p)/c+1)}, \tag{3.18}$$

while for r and s relatively prime

$$f_{rn}(\alpha, s/r) = \prod_{\nu=0}^{r-1} \frac{f_n(\alpha+s\nu/r, s)}{f_n(s\nu/r, s)}. \tag{3.19}$$

Using (3.19) and (3.18) in (3.17) it follows that for β rational of the form (3.9),

$$\Big\langle \prod_{l=1}^{n} e^{i\beta b\theta_l/2} |1+e^{i\theta_l}|^{\beta q} \Big\rangle_{\mathrm{C}\beta\mathrm{E}_{rn}} \underset{n\to\infty}{\sim} (rn)^{(\beta/2)(q^2-b^2)} A_{q,b} \tag{3.20}$$

where

$$A_{q,b} := r^{-(q^2-b^2)\beta/2} \prod_{\nu=0}^{r-1}\prod_{p=0}^{s-1} \frac{G((q+b)/r+\nu/r-p/s+1)G^2((q-b)/r+\nu/r-p/s+1)}{G(2q/r+\nu/r-p/s+1)G(\nu/r-p/s+1)}. \tag{3.21}$$

Note that in the case $b=0$ this reduces to (3.11), (3.10) as it must.

Knowing how, from the Fisher-Hartwig formula (1.8), (1.9), to generalize from the case $R=1$, general parameters, and the case general R but $b_r = 0$ ($r = 1, \dots, R$), to the case of general parameters and general R lets us use (3.13) and (3.20) to formulate a β-generalization of the Fisher-Hartwig formula for general parameters.

Conjecture 2. *Let β be rational and of the form (3.9), and let $a(\theta)$ be as assumed for the validity of (3.7). We expect, for some range of parameters $\{b_j\}$ and $\{q_j\}$,*

$$\Big\langle \prod_{l=1}^{n} a(\theta_l) \prod_{j=1}^{R} e^{-i(\beta/2)b_j \arg e^{i(\phi_j+\pi-\theta_l)}} |e^{i\theta_l} - e^{i\phi_j}|^{q_j\beta} \Big\rangle_{\mathrm{C}\beta\mathrm{E}_n} \underset{n\to\infty}{\sim} e^{c_0 n} n^{(\beta/2)\sum_{j=1}^{R}(q_j^2-b_j^2)} \tilde{E}^{(\beta)} \tag{3.22}$$

where, with $A_{q,b}$ specified by (3.21),

$$\begin{aligned} \tilde{E}^{(\beta)} &= e^{(2/\beta)\sum_{k=1}^{\infty} k c_k c_{-k}} \prod_{r=1}^{R} e^{-(q_r+b_r)\log a_-(\theta_r)} e^{-(q_r-b_r)\log a_+(\theta_r)} \\ &\quad \times \prod_{1\le r\ne s\le R} (1-e^{i(\theta_s-\theta_r)})^{-\beta(q_r+b_r)(q_s-b_s)/2} \prod_{j=1}^{R} A_{q_j,b_j}. \end{aligned} \tag{3.23}$$

4 Fisher-Hartwig asymptotics for averages over the orthogonal and symplectic groups

A problem in mathematical physics which, along with the Ising correlations, motivated the Fisher-Hartwig formula (1.8) is the impenetrable Bose gas on a circle. If the circle has circumference length L, it was shown by Lenard [19] that the ground state density matrix $\rho_{N+1}^C(x)$ has the Toeplitz determinant form

$$\begin{aligned} \rho_{N+1}^C(x;0) &= \frac{1}{L} \det[a_{j-k}^C(x)]_{j,k=1,\dots,N} \\ a_l^C(x) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{2\pi i x/L} + e^{i\theta}||1+e^{i\theta}| e^{-il\theta}\, d\theta. \end{aligned} \tag{4.1}$$

The symbol in (4.1) is of the form (1.6) with

$$a(\theta)=1,\quad R=2,\quad a_1=a_2=\frac{1}{2},\quad b_1=b_2=0,\quad \theta_1=0,\quad \theta_2=2\pi x/L. \tag{4.2}$$

Now a fundamental issue relating to the Bose gas is the occupation λ_0 of the zero momentum state, which quantifies the phenomenum of Bose-Einstein condensation (see e.g. [20]). In the present system, which is translationally invariant, λ_0 is related to the density matrix by the simple formula

$$\lambda_0=\int_0^L \rho^C_{N+1}(x;0)\,dx = L\int_0^1 \rho^C_{N+1}(LX;0)\,dX. \tag{4.3}$$

For fixed $0<X<1$ one thus seeks the $L\to\infty$ asymptotic form of $\rho^C_{N+1}(LX;0)$. In an unpublished work as of 1968, made available to the authors of [6] and subsequently published in 1972 [21], Lenard obtained for the sought expansion

$$\rho^C_{N+1}(LX;0)\sim \rho_0\frac{G^4(3/2)}{\sqrt{2\pi}}\Big(\frac{\pi}{N\sin(\pi X)}\Big)^{1/2} \tag{4.4}$$

where ρ_0 denotes the bulk density. Lenard obtained (4.4) as an upper bound, which soon after was shown to be attained by Widom [22]. Applying the Fisher-Hartwig formula (1.8) with variables (4.2) reproduces (4.4).

According to (3.1) the Toeplitz formula (4.1) can equivalently be written as the $U(N)$ average

$$\rho^C_{N+1}(x;0)=\frac{1}{L}\Big\langle\prod_{l=1}^N\Big|2\sin\Big(\frac{\pi x}{L}-\frac{\theta_l}{2}\Big)\Big|\,\Big|2\sin\frac{\theta_l}{2}\Big|\Big\rangle_{U(N)}. \tag{4.5}$$

The study of the ground state density matrices for the impenetrable Bose gas on a line of length L with Dirichlet or Neumann boundary conditions leads to formulas analogous to (4.5), only now the averages are with respect to the eigenvalue probability density functions for the classical groups $Sp(N)$ and $O^+(2N)$ respectively (see e.g. [3] for the specification of these PDFs). Thus one has [20]

$$\begin{aligned}
\rho^D_{N+1}(x;y) &= \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\Big\langle\prod_{l=1}^N\Big|2(\cos\frac{\pi x}{L}-\cos\theta_l)\Big|\Big|2(\cos\frac{\pi y}{L}-\cos\theta_l)\Big|\Big\rangle_{Sp(N)}\\
\rho^N_{N+1}(x;y) &= \frac{1}{2L}\Big\langle\prod_{l=1}^N\Big|2(\cos\frac{\pi x}{L}-\cos\theta_l)\Big|\Big|2(\cos\frac{\pi y}{L}-\cos\theta_l)\Big|\Big\rangle_{O^+(2N)}.
\end{aligned} \tag{4.6}$$

Impenetrable bosons on the interval $[0,L]$ with Dirichlet boundary conditions at $x=0$ and Neumann boundary conditions at $x=L$ also relate to a classical group. Thus from the fact that the single particle wave functions are given by

$$\phi^M_k(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi(k-1/2)x}{L},\qquad (k=1,2,\dots)$$

(the superscript M stands for "mixed") we see that the ground state wave function

$$\psi^M_0(x_1,\dots,x_N)=\frac{1}{\sqrt{N!}}\Big|\det[\phi^M_k(x_j)]_{j,k=1,\dots,N}\Big|$$

has the product form

$$\psi^M_0(x_1,\dots,x_N)=\frac{1}{\sqrt{N!}}\Big(\frac{1}{2\sqrt{L}}\Big)^N\prod_{l=1}^N 2\sin(\pi x_l/2L)\prod_{1\le j<k\le N}2|\cos\pi x_k/L-\cos\pi x_j/L|.$$

The square of this quantity coincides with the eigenvalue PDF of the classical group $O^+(2N+1)$ with $\theta = \pi x/L$ (for this we ignore the fixed eigenvalue at $\theta = 0$). From this fact, as in the derivation of (4.6) detailed in [20], it follows that

$$\rho^M_{N+1}(x;y) = \frac{2}{L}\sin\frac{\pi x}{2L}\sin\frac{\pi y}{2L}\Big\langle\prod_{l=1}^N\Big|2(\cos\frac{\pi x}{L}-\cos\theta_l)\Big|\Big|2(\cos\frac{\pi y}{L}-\cos\theta_l)\Big|\Big\rangle_{O^+(2N+1)}. \tag{4.7}$$

Using a combination of analytic calculations based on the Selberg correlation integral [28], and physical arguments based on log-gas analogies, the large N, fixed x/L, y/L, N/L limit of the density matrices (4.6) was computed in [9] to be equal to

$$\rho^D_{N+1}(x;y) \sim \rho^N_{N+1}(x;y) \sim \rho\frac{G^4(3/2)}{\sqrt{2N}}\frac{(X(1-X))^{1/8}(Y(1-Y))^{1/8}}{|X-Y|^{1/2}}\Big|_{\substack{X=(1+\cos\pi x/L)/2\\ Y=(1+\cos\pi y/L)/2}}. \tag{4.8}$$

Here we will show how recent rigorous asymptotic analysis [10] of Toeplitz + Hankel determinants (1.11) with Fisher-Hartwig type symbols can be used to prove that $\rho^M_{N+1}(x;y)$ exhibits the same asymptotic form (4.8). We will also show how a result of [10] can be used to confirm the asymptotic form of a more general class of averages over $O^+(2N+1)$ which can be deduced from a conjecture in [9], and how this conjecture in turn can be used to predict analogous asymptotics in the case of averages over $Sp(N)$ and $O^+(2N)$.

To begin we require a simple to verify identity noted in [23].

Lemma 1. *Suppose $g(\theta) = g(-\theta)$ and set $g_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ij\theta}\,d\theta$. We have*

$$\det[g_{j-k}+g_{j+k+1}]_{j,k=0,\dots,N-1} = \Big\langle\prod_{j=1}^N g(\theta_j)\Big\rangle_{O^-(2N+1)} = \Big\langle\prod_{j=1}^N g(\pi-\theta_j))\Big\rangle_{O^+(2N+1)}. \tag{4.9}$$

Note that by the assumption on $g(\theta)$ the matrix in (4.9) is symmetric. Also, the average in (4.7) is an even function of θ_1 and corresponds to the special case

$$\begin{aligned} g(\theta) &= 2|\cos\frac{\pi x}{L}-\cos\theta|\,2|\cos\frac{\pi y}{L}-\cos\theta| \\ &= \Big(|2-2\cos(\theta-\frac{\pi x}{L})||2-2\cos(\theta+\frac{\pi x}{L})||2-2\cos(\theta-\frac{\pi y}{L})||2-2\cos(\theta+\frac{\pi y}{L})|\Big)^{1/2} \end{aligned} \tag{4.10}$$

of (4.9). We observe that (4.10) is an example of a symbol of the form (1.6). Fortunately, recent rigorous works [30, 10] have determined the asymptotic form of the Hankel + Toeplitz determinant in (4.9) for all symbols (1.6), with the restriction that for $g(\theta)$ even (the case of interest in relation to (4.9)), $\theta_r \neq 0, \pm\pi$. Let us recall the result of [10, Thm. 6.1], simplified so that it relates to the even case of (1.6) with each $b_r = 0$.

Theorem 1. *Let*

$$\log g(\theta) = \log a(\theta) + \sum_{r=1}^R a_r\Big(\log(2|\cos\theta-\cos\theta_r|)\Big), \tag{4.11}$$

where $a(\theta)$ is an even periodic function with the property that the Fourier expansion of its logarithm (1.7) satisfies (1.4), together with some technical assumptions (for the latter, which may not be necessary, see [10]). We have

$$\det[g_{j-k}+g_{j+k+1}]_{j,k=0,\dots,N-1} \sim e^{(N+\sum_{j=1}^R a_j)c_0}(2N)^{\sum_{r=1}^R a_r^2}E \tag{4.12}$$

where

$$E = \prod_{r=1}^{R} \frac{G^2(1+a_r)}{G(1+2a_r)} e^{\frac{1}{2}\sum_{k=1}^{\infty} kc_k^2 + \sum_{k=1}^{\infty} c_{2k-1}} e^{-\sum_{j=1}^{N} a_j \log a(\theta_j)}$$
$$\times \prod_{r=1}^{R} \frac{|1-e^{i\theta_r}|^{a_r}}{|1+e^{i\theta_r}|^{a_r}|1-e^{2i\theta_r}|^{a_r^2}} \prod_{1\le r<s\le R} \Big(|1-e^{i(\theta_r-\theta_s)}||1-e^{i(\theta_r+\theta_s)}|\Big)^{-2a_r a_s}. \tag{4.13}$$

Recalling (4.9) it follows from Theorem 1 that

$$\Big\langle \prod_{j=1}^{N} |2(\cos\frac{\pi x}{L} - \cos\theta_j)|\,|2(\cos\frac{\pi y}{L} - \cos\theta_j)| \Big\rangle_{O^-(2N+1)}$$
$$\sim (2N)^{1/2} G^4(3/2) \frac{|1+e^{\pi i x/L}|^{1/4}}{|1-e^{\pi i x/L}|^{3/4}} \frac{|1+e^{\pi i y/L}|^{1/4}}{|1-e^{\pi i y/L}|^{3/4}} \frac{1}{|1-e^{\pi i(x-y)/L}|^{1/2}|1-e^{\pi i(x+y)/L}|^{1/2}} \tag{4.14}$$

Substituting this in (4.7) shows

$$\rho_{N+1}^{M}(x,y) \sim \frac{(2N)^{1/2}}{2L} G^4(3/2) \frac{|1-e^{2\pi i x/L}|^{1/4}|1-e^{2\pi i y/L}|^{1/4}}{|1-e^{\pi i(x-y)/L}|^{1/2}|1-e^{\pi i(x+y)/L}|^{1/2}}$$
$$= \rho \frac{G^4(3/2)}{\sqrt{2N}} \frac{(X(1-X))^{1/8}(Y(1-Y))^{1/8}}{|X-Y|^{1/2}}\Big|_{\substack{X=(1+\cos\pi x/L)/2 \\ Y=(1+\cos\pi y/L)/2}}, \tag{4.15}$$

thus rigorously establishing the asymptotic form (4.8) derived, but not rigorously proved, in [9] for the cases of Dirichlet and Neumann boundary conditions.

The eigenvalue distributions for $Sp(N)$, $O^+(2N)$, $O^-(2N+1)$ and $O^+(2N+1)$ are proportional to

$$\prod_{l=1}^{N}(1+\cos\theta_l)^{\lambda_1}(1-\cos\theta_l)^{\lambda_2} \prod_{1\le j<k\le N}(\cos\theta_k - \cos\theta_j)^2, \qquad 0 \le \theta_l \le \pi \tag{4.16}$$

for $(\lambda_1,\lambda_2) = (1,1)$, $(0,0)$, $(1,0)$ and $(0,1)$ respectively (our convention is not to include the delta function corresponding to a fixed eigenvalue, nor the delta functions corresponding to the conjugate eigenvalues). Thus to obtain the asymptotics of the averages in (4.6) it is sufficient to obtain the asymptotics of

$$\Big\langle \prod_{l=1}^{N}\prod_{r=1}^{R} \Big(2|\cos\theta_l - \cos\phi_r|\Big)^{2a_r} \Big\rangle_{C_N(\lambda_1,\lambda_2)} \tag{4.17}$$

where $C_N(\lambda_1,\lambda_2)$ refers to the normalized form of (4.16). In the cases $(\lambda_1,\lambda_2) = (1,0)$ or $(0,1)$, due to the identity (4.9), we can read off the asymptotic form from (4.12). But for general (λ_1,λ_2) the asymptotic form of (4.17) is not included in Theorem 1. Instead, we will use a conjecture from [9] to formulate the result.

Let us first recall the conjectured asymptotic form from [9]. Define

$$H_{n,\lambda_1,\lambda_2}[f(x)] := \int_0^1 dx_1 \cdots \int_0^1 dx_n \prod_{l=1}^{n} f(x_l) x_l^{\lambda_1}(1-x_l)^{\lambda_2} \prod_{1\le j<k\le n}|x_k - x_j|^2. \tag{4.18}$$

Then the argument given in [9] predicts[2]

$$\frac{H_{n,\lambda_1,\lambda_2}[e^{h(x)}\prod_{r=1}^R |y_r - x|^{2q_r}]}{H_{n+\sum_{j=1}^R q_j,\lambda_1,\lambda_2}[1]}$$
$$\sim \exp\Big[\frac{n+\sum_{r=1}^R q_r + (\lambda_1+\lambda_2)/2}{\pi}\int_0^1 \frac{h(x)}{[x(1-x)]^{1/2}}\,dx\Big]\exp\Big[\sum_{r=1}^R(-q_r+q_r^2)\log 2n\Big]K \tag{4.19}$$

where

$$\begin{aligned} K &= \prod_{j=1}^R y_j^{-\lambda_1 q_j}(1-y_j)^{-\lambda_2 q_j}\prod_{1\le j<k\le R}|y_k-y_j|^{-2q_jq_k}e^{-(\lambda_1 h(0)+\lambda_2 h(1))/2}e^{-\sum_{r=1}^R q_r h(y_r)} \\ &\quad\times\exp\Big[\frac{1}{4\pi^2}\int_0^1 dx\,\frac{h(x)}{(x(1-x))^{1/2}}\int_0^1 dy\,\frac{h'(y)(y(1-y))^{1/2}}{x-y}\Big] \\ &\quad\times\prod_{r=1}^R(y_r(1-y_r))^{-q_r{}^2/2}\prod_{r=1}^R\frac{1}{\pi^{q_r}}\frac{G^2(q_r+1)}{G(2q_r+1)}. \end{aligned} \tag{4.20}$$

In preparation for relating this to the average (4.17) let

$$h\Big(\frac{1}{2}(1+\cos\theta)\Big) = c_0 + 2\sum_{n=1}^\infty c_n\cos n\theta. \tag{4.21}$$

We then have that

$$\frac{1}{\pi}\int_0^1\frac{h(x)}{(x(1-x))^{1/2}}\,dx = \frac{1}{\pi}\int_0^\pi h\Big(\frac{1}{2}(1+\cos\theta)\Big)\,d\theta = c_0 \tag{4.22}$$

$$\frac{1}{2}(\lambda_1 h(0)+\lambda_2 h(1)) = \frac{1}{2}(\lambda_1+\lambda_2)c_0 + \sum_{n=1}^\infty c_n(\lambda_1+(-1)^n\lambda_2) \tag{4.23}$$

while (4.21) together with the cosine expansion

$$\log(2|\cos\theta-\cos\phi|) = -\sum_{n=1}^\infty\frac{2}{n}\cos n\theta\cos n\phi$$

shows

$$\frac{1}{4\pi^2}\int_0^1 dx\,\frac{h(x)}{(x(1-x))^{1/2}}\int_0^1 dy\,\frac{h'(y)(y(1-y))^{1/2}}{x-y} = \frac{1}{2}\sum_{n=1}^\infty nc_n^2. \tag{4.24}$$

Also, as noted in [9],

$$H_{n,a,b}[1] = \frac{G(n+1+a)G(n+1+b)G(n+1+a+b)}{G(1+a)G(1+b)G(2n+1+a+b)}G(n+2). \tag{4.25}$$

Since [24]

$$\log\frac{G(n+1+a)}{G(n+1+b)} \underset{n\to\infty}{\sim} (b-a)n + \frac{a-b}{2}\log(2\pi) + \Big((a-b)n + \frac{a^2-b^2}{2}\Big)\log n + o(1) \tag{4.26}$$

[2]Unfortunately there are a number of inaccuracies in the reporting of the conjecture in [9]. The term $H_{n+\sum_{j=1}^R q_j,\lambda_1,\lambda_2}[1]$ in the denominator on the left hand side of (4.19) has mistakenly been written as $H_{n,\lambda_1,\lambda_2}[1]$ in equations (90), (94), (96) and (97); the factors $\prod_{j=1}^R y_j^{-\lambda_1 q_j}(1-y_j)^{-\lambda_2 q_j}$ are missing and should be paired with $\prod_{1\le j<k\le R}|y_k-y_j|^{-2q_jq_k}$ throughout; and the term $e^{-(\lambda_1+\lambda_2)[h(0)+h(1)]/4}$ in (96) and (97) should read $e^{-(\lambda_1 h(0)+\lambda_2 h(1))/2}$.

we deduce

$$\frac{H_{n,a,b}[1]}{H_{n+Q,a,b}[1]} \sim \frac{2^{4nQ+2Q^2+2Q(a+b)}}{(2\pi n)^Q}. \tag{4.27}$$

Finally we note that under the change of variables

$$x_l = \frac{1}{2}(1+\cos\theta_l)$$

the integrand in (4.18) contains as a factor the (unnormalized) eigenvalue probability density function (4.16). Explicitly, with $\tilde{h}(\theta) := h(\frac{1}{2}(1+\cos\theta))$ we have

$$\frac{H_{n,\lambda_1,\lambda_2}[e^{h(x)}\prod_{r=1}^R |y_r - x|^{2q_r}]}{H_{n,\lambda_1,\lambda_2}[1]}\Big|_{y_r=\frac{1}{2}(1+\cos\phi_r)} = \Big\langle \prod_{l=1}^n e^{\tilde{h}(\theta_l)} \prod_{r=1}^R \Big|\frac{1}{2}(\cos\phi_r - \cos\theta_l)\Big|^{2q_r}\Big\rangle_{C_N(\lambda_1,\lambda_2)}. \tag{4.28}$$

Making use of (4.22)–(4.28) shows that the asymptotic formula (4.19) for the integral (4.18) is equivalent to an asymptotic formula generalizing Theorem 1.

Conjecture 3. *Let $\log a(\theta)$ have the Fourier expansion (1.7), with coefficients satisfying (1.4). We expect that for $N \to \infty$*

$$\Big\langle \prod_{l=1}^N a(\theta_l) \prod_{r=1}^R |2(\cos\phi_r - \cos\theta_l)|^{2a_r}\Big\rangle_{C_N(\lambda_1+1/2,\lambda_2+1/2)} \sim e^{(N+\sum_{r=1}^R a_r)c_0}(2N)^{\sum_{r=1}^R a_r^2}\tilde{K} \tag{4.29}$$

where, with E specified by (4.13),

$$\tilde{K} = \prod_{r=1}^R \frac{1}{|1+e^{i\phi_r}|^{2(\lambda_1-1)a_r}|1+e^{i\phi_r}|^{2\lambda_2 a_r}} e^{-\sum_{n=1}^\infty c_n(\lambda_1-1+(-1)^n\lambda_2)}E. \tag{4.30}$$

We can apply some checks to (4.19). As already remarked, with $(\lambda_1,\lambda_2)=(1,0)$ the probability density function $C_N(\lambda_1,\lambda_2)$ coincides with the eigenvalue probability density function for $O^-(2N+1)$, and (4.29) must coincide with (4.12), as indeed it does. Also, changing variables $\theta_l \mapsto \pi - \theta_l$ and interchanging λ_1 and λ_2 leaves (4.16) invariant, and thus the average (4.17) invariant if we also put $\phi_r \mapsto \pi - \phi_r$, $a(\theta) \mapsto a(\pi-\theta)$ (and thus $c_n \mapsto (-1)^n c_n$). Recalling the definition (4.13) of E we see that (4.30) exhibits this symmetry. Another check follows from a factorization identity, relating an average over the unitary group to a product of averages over the orthogonal and symplectic groups [25, 23, 26].

Proposition 1. *With $g(\theta) = g(-\theta)$ we have*

$$\Big\langle \prod_{l=1}^{2N+1} g(\theta_l)\Big\rangle_{U(2N+1)} = \Big\langle \prod_{l=1}^{N+1} g(\theta_l)\Big\rangle_{O^+(2N+2)}\Big\langle \prod_{l=1}^{N} g(\theta_l)\Big\rangle_{Sp(N)} \tag{4.31}$$

With

$$g(\theta) = a(\theta)\prod_{r=1}^R (2|\cos\theta - \cos\phi_r|)^{2a_r}$$

in (4.31) we see that the conjectured asymptotic form (4.29) for the right hand side is consistent with the Fisher-Hartwig formula (1.8).

The identity (4.31) is also of interest for providing an exact formula for the product of the density matrix in Dirichlet boundary conditions and in Neumann boundary conditions. Thus recalling (4.6) we see from (4.31) that

$$\frac{1}{L^2}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\Big\langle \prod_{l=1}^{2N+1}\Big|2(\cos\frac{\pi x}{L} - \cos\theta_l)\Big|\Big|2(\cos\frac{\pi x}{L} - \cos\theta_l)\Big|\Big\rangle_{U(2N+1)} = \rho_{N+2}^N(x,y)\rho_{N+1}^D(x,y).$$

5 Impenetrable bosons in a harmonic trap and random matrix averages over the GUE and LUE

From a physical viewpoint the most relevant setting for the impenetrable Bose gas is confinement by a harmonic potential (see [20, 27] and references therein). Then the ground state wave function ψ_0^H is proportional to

$$\prod_{l=1}^N e^{-x_l^2/2} \prod_{1\le j<k\le N} |x_k - x_j|,$$

and $|\psi_0^H|^2$ is identical to the eigenvalue probability density function for the Gaussian unitary ensemble of complex Hermitian matrices. The combination of log-gas arguments and analytic calculation based on the Selberg correlation integral used to analyze (4.18) was used in [27] to analyze the asymptotic form of

$$e^{-\sum_{j=1}^R 2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[\prod_{r=1}^R |x-y_r|^{2q_r}]}{G_{N+\sum_{r=1}^R q_r,\sqrt{2N}}[1]} \tag{5.1}$$

where

$$G_{N,a}[f(x)] := \int_{-\infty}^\infty dx_1 \cdots \int_{-\infty}^\infty dx_N \prod_{l=1}^N f(x_l) e^{-a^2x_l^2} \prod_{1\le j<k\le N} |x_k - x_j|^2, \tag{5.2}$$

in the special case $R = 2$, $q_1 = q_2 = 1/2$ which specifies the ground state density matrix. As our first point of interest we will generalize this calculation to general R and q_r ($q_r > -1/2$).

The log-gas perspective [18] suggests the factorization

$$\prod_{1\le j<k\le R} |y_j - y_k|^{2q_jq_k} e^{-\sum_{r=1}^R 2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[\prod_{r=1}^R |x-y_r|^{2q_r}]}{G_{N+\sum_{r=1}^R q_r,\sqrt{2N}}[1]} \underset{N\to\infty}{\sim} \prod_{r=1}^R e^{-2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[|x-y_r|^{2q_r}]}{G_{N+q_r,\sqrt{2N}}[1]}. \tag{5.3}$$

Next, from the theory of Selberg correlation integrals [28], for $q_r \in \mathbb{Z}_{\ge 0}$ we have the duality formula [29]

$$\frac{G_{N,a}[(x-y_r)^{2q_r}]}{G_{N,a}[1]} = \frac{G_{2q_r,a}[(y_r+ix)^N]}{G_{2q_r,a}[1]}. \tag{5.4}$$

It is a fairly straight forward exercise, detailed in [29, section 5.3] for a related problem, to use the saddle point method to compute the large N expansion of the integral on the right hand side of (5.4). One finds

$$e^{-2Nq_r x_r^2} G_{2q_r,\sqrt{2N}}[(y_r+ix)^N] \underset{N\to\infty}{\sim} \binom{2q_r}{q_r} (G_{q_r,1}[1])^2 e^{-q_rN} 2^{-2q_rN} (4N)^{-q_r^2} (1-y_r^2)^{q_r^2/2}. \tag{5.5}$$

Also, we know (see e.g. [3]) the exact evaluation

$$G_{n,1}[1] = 2^{-n(n-1)/2} \pi^{n/2} G(n+2), \tag{5.6}$$

which together with the asymptotic expansion (4.26) implies

$$\frac{G_{N,\sqrt{2N}}[1]}{G_{N+q,\sqrt{2N}}[1]} \underset{N\to\infty}{\sim} \frac{2^{2Nq+q^2-q}}{\pi^q} e^{qN} N^{-q}. \tag{5.7}$$

Combining (5.4)–(5.7) gives the asymptotic formula

$$e^{-2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[|x-y_r|^{2q_r}]}{G_{N+q_r,\sqrt{2N}}[1]} \underset{N\to\infty}{\sim} \frac{G^2(q_r+1)}{G(2q_r+1)} \frac{2^{2q_r^2-q_r}}{\pi^{q_r}} N^{q_r^2-q_r} (1-y_r^2)^{q_r^2/2} \tag{5.8}$$

which we have proved for $q_r \in \mathbb{Z}_{\ge 0}$, and conjecture as being valid for all $q_r > -1/2$. This same result can also be deduced from results in [31]. Substituting this in (5.3) gives the sought asymptotic form of (5.1).

Conjecture 4. *Let $q_r > -1/2$, and let $G_{N,a}[f]$ be given by (5.2). We expect*

$$e^{-\sum_{r=1}^R 2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[\prod_{r=1}^R |x-y_r|^{2q_r}]}{G_{N+\sum_{r=1}^R q_r,\sqrt{2N}}[1]}$$
$$\underset{N\to\infty}{\sim} \prod_{1\le j<k\le R} |y_j - y_k|^{-2q_jq_k} \prod_{r=1}^R \frac{G^2(q_r+1)}{G(2q_r+1)} \frac{2^{2q_r^2-q_r}}{\pi^{q_r}} N^{q_r^2-q_r}(1-y_r^2)^{q_r^2/2}. \tag{5.9}$$

An extension of (5.9) can also be formulated. Let $a(x)$ be analytic on $[-1,1]$. Then it has rigorously been proved that [16]

$$\frac{G_{N,\sqrt{2N}}[e^{a(x)}]}{G_{N,\sqrt{2N}}[1]}$$
$$\underset{N\to\infty}{\sim} \exp\Big(\frac{2N}{\pi}\int_{-1}^1 a(x)\sqrt{1-x^2}\,dx\Big)\exp\Big(\frac{1}{4\pi^2}\int_{-1}^1 dx\,\frac{a(x)}{(1-x^2)^{1/2}}\int_{-1}^1 dy\,\frac{a'(y)(1-y^2)^{1/2}}{x-y}\Big) \tag{5.10}$$

The structure of (4.19) in the case $\lambda_1 = \lambda_2 = 0$ suggests how (5.10) can be combined with (5.9) to generalize the latter.

Conjecture 5. *Let $a(x)$ be analytic on $[-1,1]$. It is expected that*

$$e^{-\sum_{r=1}^R 2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[e^{a(x)}\prod_{r=1}^R |x-y_r|^{2q_r}]}{G_{N+\sum_{r=1}^R q_r,\sqrt{2N}}[1]}$$
$$\underset{N\to\infty}{\sim} \Big(\mathrm{RHS}\,(5.10)\Big)\Big|_{N\mapsto N+\sum_{r=1}^R q_r} \Big(\mathrm{RHS}\,(5.9)\Big) e^{-\sum_{r=1}^R q_r a(y_r)}. \tag{5.11}$$

We remark that in the special case $a(x) = kx$, Conjecture 5 can be reduced to Conjecture 4. To see this, use completion of squares to note

$$G_{N,\sqrt{2N}}\Big[e^{kx}\prod_{r=1}^R \Big|x-y_r\Big|^{2q_r}\Big] = e^{k^2/8} G_{N,\sqrt{2N}}\Big[\prod_{r=1}^R \Big|x+\frac{k}{2N}-y_r\Big|^{2q_r}\Big].$$

According to Conjecture 4 we have

$$e^{-\sum_{r=1}^R 2Nq_r(y_r-k/4N)^2} \frac{G_{N,\sqrt{2N}}[e^{kx}\prod_{r=1}^R |x+\frac{k}{2N}-y_r|^{2q_r}]}{G_{N+\sum_{r=1}^R q_r,\sqrt{2N}}[1]} \underset{N\to\infty}{\sim} \mathrm{RHS}\,(5.10)$$

and thus

$$e^{-\sum_{r=1}^R 2Nq_r y_r^2} \frac{G_{N,\sqrt{2N}}[e^{kx}\prod_{r=1}^R |x-y_r|^{2q_r}]}{G_{N+\sum_{r=1}^R q_r,\sqrt{2N}}[1]} \sim e^{k^2/8}\,\mathrm{RHS}\,(5.10)\, e^{-\sum_{r=1}^R q_r y_r}.$$

Since (4.24) gives

$$\frac{1}{4\pi^2}\int_{-1}^1 dx\,\frac{a(x)}{(1-x^2)^{1/2}}\int_{-1}^1 dy\,\frac{a'(y)(1-y^2)^{1/2}}{x-y}\Big|_{a(x)=kx} = \frac{k^2}{8}$$

this is in agreement with (5.11).

Let us now turn our attention to a variation of the impenetrable Bose gas in a harmonic well, which also has the features of being related to a random matrix ensemble. In reduced units, the Hamiltonian for the system is

$$H = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^N \Big(a'(a'-1)\frac{1}{x_j^2} + x_j^2\Big), \qquad x_j > 0. \tag{5.12}$$

Thus in addition to the harmonic well, the particles are restricted to the half line by a repulsive potential (requiring $a' > 1$) at the origin proportional to $1/r^2$. This is the non-interacting case of the so called

type B Calogero-Sutherland Hamiltonian [32], for which the interacting case has $1/r^2$ pair repulsion. The ground state wave function for (5.12) is proportional to

$$\prod_{l=1}^{N} e^{-x_l^2/2}(x_l^2)^{a'/2} \prod_{1\le j<k\le N} |x_k^2 - x_j^2|. \tag{5.13}$$

We recognize the square of the ground state wave function as being identical to the probability density function for the singular values of $n \times N$ complex Gaussian matrices with $a' = n - N + 1/2$ (see e.g. [3, Ch. 2]. Changing variables $x_l^2 \mapsto x_l$ this is refered to as the Laguerre unitary ensemble. The problem of computing the asymptotic form of the density matrix for this system suggests analyzing the asymptotic form of the more general quantity

$$\prod_{r=1}^{R} y_r^{2a' q_r} e^{-4Nq_r y_r^2} \frac{L_N[\prod_{r=1}^{R} |x^2 - y_r^2|^{2q_r}]}{L_{N+\sum_{r=1}^{R} q_r}[1]} \tag{5.14}$$

where

$$L_N[f] := \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{l=1}^{N} f(x_l) x_l^{2a'} e^{-4Nx_l^2} \prod_{1\le j<k\le N} |x_k^2 - x_j^2|^2, \tag{5.15}$$

in analogy with (5.1).

From a log-gas perspective, the integrand in (5.15) corresponds to a one-component system interacting on the half line $x > 0$, subject to a one-body confining potential $2Nx^2 - (a' - 1/2)\log x$. In addition to the electrostatic energy $-\log|x - x'|$ at the point x due to the interaction with a charge at x', there is also a term $-\log|x + x'|$ due to the interaction with an image charge at $-x'$ (outside the system, since $x' > 0$). In keeping with the image charge interpretation, for each charge at x one requires a term $-\frac{1}{2}\log|2x|$ to account for the interaction between a charge and its own image (the factor of $1/2$ is because this energy is shared between the charge and its image, the latter being outside the system). From this viewpoint we can interpret

$$\prod_{1\le j<k\le R} |y_k^2 - y_j^2|^{2q_j q_k} \prod_{r=1}^{R} (2y_r)^{q_r^2} e^{-q_r 4N y_r^2} |y_r|^{(2a'-1)q_r} \frac{L_N[\prod_{r=1}^{R} |x^2 - y_r^2|^{2q_r}]}{2^{\sum_{r=1}^{R} q_r} L_{N+\sum_{r=1}^{R} q_r}[1]}$$

as a ratio of partition functions for log-gas systems, and analogous to (5.3) we expect the factorization into

$$\prod_{r=1}^{R} (2y_j)^{q_j^2} e^{-q_r 4N y_r^2} |y_r|^{(2a'-1)q_r} \frac{L_N[|x^2 - y_r^2|^{2q_r}]}{2^{q_r} L_{N+q_r}[1]} \tag{5.16}$$

for $N \to \infty$.

To analyze (5.16) in the limit $N \to \infty$ we make the change of variables $x_l^2 \mapsto x_l$ and introduce

$$\tilde{L}_{N,c}[f] := \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{l=1}^{N} f(x_l) x_l^{a'-1/2} e^{-cx_l} \prod_{1\le j<k\le N} (x_k - x_j)^2 \tag{5.17}$$

so that it reads

$$\prod_{r=1}^{R} (2y_r)^{q_r^2} e^{-q_r 4N y_r^2} y_r^{(2a'-1)q_r} \frac{\tilde{L}_{N,4N}[|x - y_r^2|^{2q_r}]}{\tilde{L}_{N+q_r,4N}[1]}. \tag{5.18}$$

To proceed further, we use the fact that for $q \in \mathbb{Z}_{\geq 0}$ we have the duality formula [33]

$$\begin{aligned}\frac{\tilde{L}_{N,c}[|x-t|^{2q}]}{\tilde{L}_{N,c}[1]\Big|_{a\mapsto a+2q}} &= \frac{1}{M_{2q}(a,N)} \\ &\times \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_{2q} \prod_{l=1}^{2q} e^{\pi i x_l (a-N)} |1+e^{2\pi i x_l}|^{a+N} e^{-cte^{2\pi i x_l}} \prod_{1\leq j<k\leq 2q} |e^{2\pi i x_k} - e^{2\pi i x_j}|^2\end{aligned} \tag{5.19}$$

where on the right hand side $a = a' - 1/2$ and

$$\begin{aligned} M_n(a,b) &:= \int_{-1/2}^{1/2} dx_1 \cdots \int_{-1/2}^{1/2} dx_n \prod_{l=1}^{n} e^{\pi i x_l (a-b)} |1+e^{2\pi i x_l}|^{a+b} \prod_{1\leq j<k\leq n} |e^{2\pi i x_k} - e^{2\pi i x_j}|^2 \\ &= \frac{G(n+1+a+b)}{G(1+a+b)} \frac{G(1+a)}{G(n+1+a)} \frac{G(1+b)}{G(n+1+b)} G(n+2) \end{aligned} \tag{5.20}$$

(for the last equality see e.g. [3]). If we suppose temporarily that $a \in Z_{\geq 0}$, the right hand side of (5.19) with $c = 4N$ can be written as the contour integral

$$\frac{1}{M_{2q}(a,-N)} \int_{\mathcal{C}} \frac{dz_1}{2\pi i z_1} \cdots \int_{\mathcal{C}} \frac{dz_{2q}}{2\pi i z_{2q}} \prod_{l=1}^{2q} (1+z_l)^a (1+1/z_l)^N e^{-4Ntz_l} \prod_{1\leq j<k\leq 2q} (z_k - z_j)(1/z_k - 1/z_j) \tag{5.21}$$

where $\mathcal{C}$ is any simple closed contour which encircles the origin. To analyze this for $N \to \infty$, following [33] where the case $q = 1$ was considered, we note the N-dependent terms in the integrand have a stationary point when

$$z = z_{\pm} := -\frac{1}{2} \pm i\frac{1}{2}(1/t - 1)^{1/2}. \tag{5.22}$$

By deforming the contour $\mathcal{C}$ to pass through z_+ for q of the integrations, and to pass through z_- for the remaining q integrations, we readily deduce from the representation (5.21) of (5.19) that

$$\begin{aligned} & e^{-q_r 4N y_r^2} y_r^{(2a'-1)q_r} (2y_r)^{q_r^2} \tilde{L}_{N,4N}[|x - y_r^2|^{2q_r}] \\ &= \frac{\tilde{L}_{N,4N}[1]|_{a'\mapsto a'+2q_r}}{M_{2q_r}(a'-1/2,N)} e^{-q_r 4N y_r^2} y_r^{(2a'-1)q_r} (2y_r)^{q_r^2} \\ &\times \binom{2q_r}{q_r} e^{-4N y_r^2 q_r (z_+ + z_-) + N q_r \log|1+1/z_+|^2} \frac{1}{|z_+|^{4q_r^2}} |z_+ - z_-|^{2q_r^2} |1+z_+|^{q_r(2a'-1)} \Big(\frac{1}{2\pi}\Big)^{2q_r} \\ &\times \Big| \frac{N}{2} \Big(\frac{1}{z_+^2} - \frac{1}{(1+z_+)^2} \Big) \Big|^{-q_r^2} (G_{q_r}[1])^2. \end{aligned} \tag{5.23}$$

Now, with $t = y_r^2$ in (5.22)

$$\begin{aligned} & z_+ + z_- = -1, \qquad |1 + \frac{1}{z_+}|^2 = 1, \qquad |z_+ - z_-|^2 = \Big(\frac{1}{y_r^2} - 1\Big) \\ & |1+z_+|^2 = |z_+|^2 = \frac{1}{4y_r^2}, \qquad \Big| \frac{1}{z_+^2} - \frac{1}{(1+z_+)^2} \Big| = 16 y_r^4 \Big(\frac{1}{y_r^2} - 1 \Big)^{1/2} \end{aligned} \tag{5.24}$$

so the right hand side of (5.23) simplifies to

$$\frac{\tilde{L}_{N,4N}[1]|_{a'\mapsto a'+2q_r}}{M_{2q_r}(a'-1/2,N)} N^{-q_r^2} \Big(\frac{1}{2\pi}\Big)^{2q_r} \binom{2q_r}{q_r} 2^{2q_r^2} 2^{-q_r(2a'-1)} (G_{q_r}[1])^2 (1-y_r^2)^{q_r^2/2}. \tag{5.25}$$

Furthermore we know (see e.g. [3])

$$\tilde{L}_{N,c}[1] = c^{-N^2 - N(a'-1/2)} \frac{G(N+2) G(a'+N+1/2)}{G(a'+1/2)}, \tag{5.26}$$

and making use too of (5.22) it follows from the asymptotic expansion (4.26) that

$$\frac{\tilde{L}_{N,4N}[1]|_{a'\mapsto a'+2q_r}}{\tilde{L}_{N+q_r,4N}[1]M_{2q_r}(a'-1/2,N)}N^{-q_r^2} \sim 2^{2(q_r^2+q_r(a'-1/2))}N^{q_r^2-q_r}. \tag{5.27}$$

Substituting (5.27) in (5.25), evaluating $G_{q_r}[1]$ therein according to (5.6) and simplifying we obtain the $N \to \infty$ expansion

$$e^{-q_r 4Ny_r^2}y_r^{(2a'-1)q_r}(2y_r)^{q_r^2}\frac{L_N[|x^2-y_r^2|^{2q_r}]}{2^{q_r}L_{N+q_r}[1]} \sim \frac{G^2(q_r+1)}{G(2q_r+1)}\frac{2^{3q_r^2-q_r}}{\pi^{q_r}}N^{q_r^2-q_r}(1-y_r^2)^{q_r^2/2}, \tag{5.28}$$

proved for $q_r \in \mathbb{Z}_{\geq 0}$ and expected to be true for all $q_r > -1/2$. Substituting this in (5.16) gives, as a conjecture, the sought asymptotic form of (5.14).

Conjecture 6. *For $N \to \infty$, and assuming $q_r > -1/2$ for each $r = 1, \dots, R$,*

$$\begin{aligned}&\prod_{r=1}^{R} y_r^{2a'q_r}e^{-4Nq_ry_r^2}\frac{L_N[\prod_{r=1}^R|x^2-y_r^2|^{2q_r}]}{L_{N+\sum_{r=1}^R q_r}[1]} \\ &\quad \sim \prod_{1\leq j<k\leq R}|y_k^2-y_j^2|^{-2q_jq_k}\prod_{r=1}^{R}\frac{G^2(q_r+1)}{G(2q_r+1)}\frac{2^{2q_r^2}}{\pi^{q_r}}N^{q_r^2-q_r}y_r^{-q_r^2+q_r}(1-y_r^2)^{q_r^2/2}.\end{aligned} \tag{5.29}$$

It is of interest to extend (5.29) in an analogous way to how (5.11) extends (5.9). First we use (4.19) with $q_r = 0$, and (5.10) to conjecture that for $a(x)$ analytic on $[0,1]$

$$\begin{aligned}\frac{L_N[e^{a(x)}]}{L_N[1]} &\sim \exp\Big(\frac{4(N+(2a'-1)/4)}{\pi}\int_0^1 a(x)\sqrt{1-x^2}\,dx\Big) \\ &\quad\times\exp\Big(\frac{1}{\pi^2}\int_0^1 dx\,\frac{a(x)}{(1-x^2)^{1/2}}\int_0^1 dy\,\frac{ya'(y)(1-y^2)^{1/2}}{x^2-y^2}\Big)e^{-a'a(0)/2}.\end{aligned} \tag{5.30}$$

Combining this with (5.29) as in (5.11) gives us the LUE analogue of Conjecture 5.

Conjecture 7. *Let $a(x)$ be analytic on $[0,1]$. It is expected that*

$$\begin{aligned}&\prod_{r=1}^{R} y_r^{2a'q_r}e^{-4Nq_ry_r^2}\frac{L_N[e^{a(x)}\prod_{r=1}^R|x^2-y_r^2|^{2q_r}]}{L_{N+\sum_{r=1}^R q_r}[1]} \\ &\quad \underset{N\to\infty}{\sim}\Big(\mathrm{RHS}\,(5.30)\Big)\Big|_{N\to N+\sum_{r=1}^R q_r}\Big(\mathrm{RHS}\,(5.29)\Big)e^{-\sum_{r=1}^R q_ra(y_r)}.\end{aligned} \tag{5.31}$$

We can check the consistency of (5.11) and (5.31). For this we make use of a factorization identity analogous to Proposition 1 [26]

Proposition 2. *Let $g(\theta) = g(-\theta)$. We have*

$$\frac{G_{2N,a}[g(x)]}{G_{2N,a}[1]} = \frac{L_{N,a}^{(0)}[g(x)]}{L_{N,a}^{(0)}[1]}\frac{L_{N,a}^{(2)}[g(x)]}{L_{N,a}^{(2)}[1]} \tag{5.32}$$

where

$$L_{N,a}^{(p)}[g(x)] := \int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_N\prod_{l=1}^{N}g(x_l)|x_l|^pe^{-a^2x_l^2}\prod_{1\leq j<k\leq N}(x_k^2-x_j^2)^2.$$

Let $a(x)$ be even and choose

$$a = \sqrt{4N}, \qquad g(x) = e^{a(x)} \prod_{r=1}^{R} (x^2 - y_r^2)^{2q_r}.$$

According to Conjecture 5,

$$\begin{aligned} & e^{-2a^2 \sum_{r=1}^R q_r y_r^2} \frac{G_{2N,a}[g(x)]}{G_{2(N+\sum_{r=1}^R q_r),a}[1]} \\ & \underset{N\to\infty}{\sim} \exp\Big(\frac{4}{\pi}(2N + 2\sum_{r=1}^R q_r) \int_0^1 a(x)\sqrt{1-x^2}\, dx\Big) \\ & \times \exp\Big(\frac{1}{\pi^2}\int_0^1 dx\, \frac{a(x)}{(1-x^2)^{1/2}} \int_0^1 dy\, \frac{ya'(y)(1-y^2)^{1/2}}{x^2-y^2}\Big) \\ & \times \prod_{1\le j<k\le R} |y_j^2 - y_k^2|^{-4q_j q_k} \prod_{r=1}^R (1-y_r^2)^{q_r^2} |2y_r|^{-2q_r^2} \\ & \times \Big(\prod_{r=1}^R \frac{G^2(q_r+1)}{G(2q_r+1)} \frac{2^{2q_r^2-q_r}}{\pi^{q_r}} (2N)^{q_r^2-q_r}\Big)^2 e^{-2\sum_{r=1}^R q_r a(y_r)}. \end{aligned} \tag{5.33}$$

For the right hand side of (5.32) as implied by Conjecture 7 to be consistent with this we require

$$2^{4\sum_{r=1}^R q_r} \frac{G_{2N,\sqrt{4N}}[1]}{L_N[1]|_{a'=0} L_N[1]|_{a'=1}} \sim \frac{G_{2(N+\sum_{r=1}^R q_r),\sqrt{4N}}[1]}{L_{N+\sum_{r=1}^R q_r}[1]|_{a'=0} L_{N+\sum_{r=1}^R q_r}[1]|_{a'=1}}. \tag{5.34}$$

But the method of derivation of (5.32) given in [26] shows that for general n,

$$\frac{G_{2n,\sqrt{4n}}[1]}{L_n|_{a'=0} L_n|_{a'=1}} = 2^{2n} \frac{(2n)!}{(n!)^2} \sim \frac{2^{4n}}{(\pi n)^{1/2}},$$

verifying (5.34).

Let us now apply Conjecture 7 to the calculation of the density matrix $\rho_{N+1}^L(x,y)$ for the state (5.13) with $N+1$ particles,

$$\begin{aligned} \rho_{N+1}^L(x,y) &:= \frac{N+1}{C_{N+1}} e^{-x^2/2 - y^2/2} (xy)^{a'} \\ & \times \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \prod_{l=1}^N x_l^{2a'} e^{-x_l^2} |x^2 - x_l^2||y^2 - y_l^2| \prod_{1\le j<k\le N} (x_k^2 - x_j^2)^2 \end{aligned} \tag{5.35}$$

where

$$C_{N+1} := \int_0^\infty dx_1 \cdots \int_0^\infty dx_{N+1} \prod_{l=1}^{N+1} x_l^{2a'} e^{-x_l^2} \prod_{1\le j<k\le N+1} (x_k^2 - x_j^2)^2.$$

In terms of the average (5.15) we thus have

$$2\sqrt{N}\rho_{N+1}^L(2\sqrt{N}X, 2\sqrt{N}Y) = (N+1) e^{-2NX^2 - 2NY^2} (XY)^{a'} \frac{L_N[\prod_{l=1}^N |x^2 - X^2||x^2 - Y^2|]}{L_{N+1}[1]}.$$

On the right hand side we can apply Conjecture 7 with $R = 2$, $q_1 = q_2 = 1/2$ and so obtain the asymptotic form

$$2\sqrt{N}\rho_{N+1}^L(2\sqrt{N}X, 2\sqrt{N}Y) \sim 2\sqrt{N} \frac{G^4(3/2)}{\pi} \frac{(XY)^{1/4}}{|X^2 - Y^2|^{1/2}} (1-X^2)^{1/8} (1-Y^2)^{1/8}. \tag{5.36}$$

The asymptotic form (5.36) can in turn be used to specify the occupations λ_j of the low-lying effective single particle states ϕ_j, which by definition satisfy the eigenvalue equation

$$\int \rho_N(x,y)\phi_j(y)\,dy = \lambda_j \phi_j(x). \tag{5.37}$$

Thus, with $x = 2\sqrt{N}X$, $y = 2\sqrt{N}Y$ and j fixed, introducing the scaled effective single particle states [34, 20]

$$(4N)^{1/2}\phi_j(x) \mapsto \varphi_j(X),$$

substituting (5.38) and using the fact that $\rho_N^L(x,y)$ is supported on $x, y \in [0, 2\sqrt{N}]$ we obtain the explicit integral equation

$$2\int_0^1 \frac{X^{1/4}(1-X^2)^{1/8}\varphi_j(X)}{|X^2-Y^2|^{1/2}}\,dX = \bar{\lambda}_j \frac{\varphi_j(Y)}{Y^{1/4}(1-Y^2)^{1/8}} \tag{5.38}$$

where

$$\lambda_j = \sqrt{N}\frac{G^4(3/2)}{\pi}\bar{\lambda}_j. \tag{5.39}$$

We see immediately that the occupations of the low-lying effective single particle states are proportional to $\sqrt{N}$, as has been found for the impenetrable Bose gas in periodic boundary conditions [19, 20], in a harmonic trap [34, 27] and in Dirichlet and Neumann boundary conditions [9]. An appropriate analysis similar to that undertaken in [27, Appendix B] gives the same upper bound on $\bar{\lambda}_0$ as found for the same quantity in the case of the harmonic trap [27], but a detailed analysis of (5.38) remains.

6 Concluding remarks

6.1 Universal form for Hankel asymptotics

Analogous to (3.1), Hankel determinants are related to log-gas partition functions according to the formula

$$\begin{aligned} &\det[a_{j+k}]_{j,k=0,\dots,n-1} \\ &\quad = \frac{1}{n!}\int_{-\infty}^{\infty} dx_1\, e^{-nV(x_1)} \cdots \int_{-\infty}^{\infty} dx_n\, e^{-nV(x_n)} \prod_{l=1}^n a(x_l) \prod_{1\le j<k\le n}(x_k-x_j)^2 \\ &\quad =: A_n(e^{-nV(x)})[a(x)] \end{aligned} \tag{6.1}$$

where

$$a_p = \int_{-\infty}^{\infty} a(x) x^p e^{-nV(x)}\,dx.$$

For $V(x)$ an even degree polynomial independent of n with positive leading coefficient and no real zeros, it was proved by Johansson [16] that

$$\begin{aligned} &\frac{A_n(e^{-nV(x)})[e^{a(x)}]}{A_n(e^{-nV(x)})[1]} \\ &\quad \underset{n\to\infty}{\sim} \exp\Big(n\int_{c_1}^{c_2} a(x)\rho(x)\,dx\Big)\exp\Big(\frac{1}{4\pi^2}\int_{c_1}^{c_2} dx \frac{a(x)}{\sqrt{(x-c_1)(c_2-x)}}\int_{c_1}^{c_2} dy \frac{a'(y)\sqrt{(y-c_1)(c_2-y)}}{x-y}\Big). \end{aligned} \tag{6.2}$$

Here $\rho(x)$ is the scaled density in the log-gas system corresponding to $A_n(e^{-nV(x)})[1]$, supported on $[c_1, c_2]$ and normalized so that

$$\int_{c_1}^{c_2} \rho(x)\,dx = 1.$$

The asymptotic formula (5.10) corresponds to the special case $V(x)=\frac{1}{2}x^2$, $\rho(x)=\frac{2}{\pi}\sqrt{1-x^2}$ of (6.2). To extend Conjecture 5 to more general V this suggests we simply write the latter in terms of $\rho(x)$.

Conjecture 8. *Under the conditions of the validity of (6.2) we expect*

$$e^{-n\sum_{r=1}^R q_r V(y_r)} \frac{A_n(e^{-nV(x)})[e^{a(x)}\prod_{j=1}^r |x-y_j|^{q_j}]}{A_{n+\sum_{j=1}^R q_j}(e^{-nV(x)})[e^{a(x)}]}$$
$$\underset{n\to\infty}{\sim} e^{-\sum_{r=1}^R q_r a(y_r)} \prod_{1\le j<k\le R} |y_k-y_j|^{-2q_jq_k} \prod_{r=1}^R \frac{G^2(q_r+1)}{G(2q_r+1)} (2\pi N)^{q_r^2-q_r} (\rho(y_r))^{q_r^2}. \qquad (6.3)$$

We remark that in the case $R=1$, $e^{a(x)}=1$, this conjecture (together with some corroborative analysis) was formulated earlier by Brézin and Hikami [31] (see also [35]).

Conjecture 7 can similarly be extended, although we work with the quantity (5.17) in favour of (5.15) so as to have a Hankel determinant interpretation according to (6.1). In the log-gas system corresponding to (5.17) one has $\rho(x)=\frac{2}{\pi x^{1/2}}(1-x)^{1/2}$. Recalling the equality between (5.16) and (5.18), and writing $y_r^2\mapsto y_r$, $a(x^{1/2})\mapsto a(x)$ we see that Conjecture 7 can be rewritten to imply

$$\prod_{r=1}^R y_r^{(a'-1/2)q_r} e^{-4Nq_ry_r} \frac{\tilde{L}_{N,4N}[e^{a(x)}\prod_{r=1}^R |x-y_r|^{2q_r}]}{\tilde{L}_{N+\sum_{r=1}^R q_r,4N}[e^{a(x)}]}$$
$$\underset{N\to\infty}{\sim} e^{-\sum_{r=1}^R q_r a(y_r)} \prod_{1\le j<k\le R} |y_k-y_j|^{-2q_jq_k} \prod_{r=1}^R \frac{G^2(q_r+1)}{G(2q_r+1)} (2\pi N)^{q_r^2-q_r} (\rho(y_r))^{q_r^2}, \qquad (6.4)$$

thus assuming the universal form (6.3) and suggesting the following analogue of (6.2) and Conjecture 8.

Conjecture 9. *Let $V(x)$ be a polynomial independent of n, with positive leading coefficient and no real zeros on $[0,\infty)$. Let*

$$\tilde{A}_n(x^\alpha e^{-nV(x)})[a(x)] := \frac{1}{n!}\int_0^\infty dx_1\, x_1^\alpha e^{-nV(x_1)} \cdots \int_0^\infty dx_n\, x_n^\alpha e^{-nV(x_n)} \prod_{l=1}^n a(x_l) \prod_{1\le j<k\le n}(x_k-x_j)^2. \qquad (6.5)$$

Analogous to (6.2) we expect that

$$\frac{\tilde{A}_n(x^\alpha e^{-nV(x)})[e^{a(x)}]}{\tilde{A}_n(x^\alpha e^{-nV(x)})[1]}$$
$$\underset{n\to\infty}{\sim} \exp\Big(n\int_0^{c_2} a(x)\rho(x)\,dx\Big)\exp\Big(\frac{1}{4\pi^2}\int_0^{c_2} dx\,\frac{a(x)}{\sqrt{x(c_2-x)}}\int_{c_1}^{c_2} dy\,\frac{a'(y)\sqrt{y(c_2-y)}}{x-y}\Big) \qquad (6.6)$$

where $\rho(x)$ is the scaled density in the log-gas corresponding to $\tilde{A}_n(x^\alpha e^{-nV(x)})[1]$, with support on $[0,c_2]$. Furthermore, with the same meaning of $\rho(x)$, we expect

$$\prod_{r=1}^R y_r^\alpha e^{-nq_rV(y_r)} \frac{\tilde{A}_n(x^\alpha e^{-nV(x)})[e^{a(x)}\prod_{j=1}^r |x-y_j|^{q_j}]}{\tilde{A}_{n+\sum_{j=1}^R q_j}(x^\alpha e^{-nV(x)})[e^{a(x)}]} \underset{n\to\infty}{\sim} \mathrm{RHS}\,(6.3). \qquad (6.7)$$

As a final comment on this point, we note that the universal form given by the right hand side of (6.3) is also exhibited by the Fisher-Hartwig formula (1.8). Thus, with $z_r := e^{i\theta_r}$ we see that

$$\frac{D_n[e^{a(\theta)}\prod_{r=1}^R |e^{i\theta}-z_r|]}{D_{n+\sum_{j=1}^R}[e^{a(\theta)}]} \underset{n\to\infty}{\sim} \mathrm{RHS}\,(6.3)\Big|_{\substack{y_r=z_r\\ \rho(y)=N/2\pi}}.$$

6.2 Further Toeplitz + Hankel structures

The identity (4.9) of Lemma 1 has counterparts for averages over $Sp(N)$ and $O^+(2N)$ [23].

Lemma 2. *Suppose $g(\theta) = g(-\theta)$, set $g_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-ij\theta}\,d\theta$, and let $C_N(\lambda_1,\lambda_2)$ refer to the normalized form of (4.16). We have*

$$\begin{aligned}
\det[a_{j-k} + a_{j+k}]_{j,k=0,\dots,N-1} &= \Big\langle \prod_{j=1}^{N} g(\theta_j) \Big\rangle_{O^+(2N)} = \Big\langle \prod_{j=1}^{N} g(\theta_j) \Big\rangle_{C_N(0,0)} \\
\det[a_{j-k} - a_{j+k+2}]_{j,k=0,\dots,N-1} &= \Big\langle \prod_{j=1}^{N} g(\theta_j) \Big\rangle_{Sp(N)} = \Big\langle \prod_{j=1}^{N} g(\theta_j) \Big\rangle_{C_N(1,1)}. \qquad (6.8)
\end{aligned}$$

Choosing $g(\theta)$ as in (4.11), Conjecture 3 gives the asymptotic behaviour of the right hand sides in (6.8), and thus the conjectured form of these Toeplitz + Hankel structures.

6.3 Fluctuation formula perspective and future directions

Let $p := p(x_1,\dots,x_N)$ be an N-dimensional probability density function. The stochastic quantity $A = \sum_{j=1}^{N} a(x_j)$, with the $\{x_j\}$ sampled from p, is referred to as a linear statistic. Its distribution $P_A(t)$ is defined by

$$P_A(t) = \Big\langle \delta\Big(t - \sum_{j=1}^{N} a(x_j)\Big)\Big\rangle_p, \qquad (6.9)$$

and taking the Fourier transform of this gives

$$\tilde{P}_A(k) = \Big\langle \prod_{j=1}^{N} e^{ika(x_j)} \Big\rangle_p. \qquad (6.10)$$

The structure of the average (6.10) is common to the averages studied in this paper. As an illustration of the content of the asymptotic formulas from this viewpoint, consider Johansson's result (3.7). Written in terms of the average (3.1) with $g(\theta) = e^{ika(\theta)}$, it reads

$$D_n^{(\beta)}[e^{ika(\theta)}] \underset{n\to\infty}{\sim} e^{ikc_0 n} e^{-(2/\beta)k^2 \sum_{n=1}^{\infty} nc_n c_{-n}} \qquad (6.11)$$

where $\{c_n\}_{n=0,\pm1,\dots}$ are the Fourier coefficients in the expansion of $a(\theta)$,

$$a(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}. \qquad (6.12)$$

A key feature of the exponents in the exponentials on the right hand side of (6.11) is that they form a quadratic polynomial in k. Thus substituting this in (6.9) and taking the inverse transform gives the Gaussian distribution

$$P_A(t) \underset{n\to\infty}{\sim} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-(t-\mu)^2/2\sigma^2} \qquad (6.13)$$

with

$$\mu = nc_0, \qquad \sigma^2 = \frac{4}{\beta}\sum_{p=1}^{\infty} pc_p c_{-p}. \qquad (6.14)$$

As noted by Johansson [5], in the case $\beta = 2$ this gives a Gaussian fluctuation formula interpetation of Szegö's theorem. A perculiar feature is that although the mean is proportional to n, the variance is $O(1)$, so fluctuations are strongly suppressed. It is formulas of the type (6.13), (6.14) which led to the

successful theoretical explanation of the phenomenom of universal conductance fluctuations in mesoscopic wires (see e.g. [36]), in which the conductance — an order N quantity — is written as a linear statistic of certain eigenvalues and is shown to have $O(1)$ fluctuations with variance given by an analytic formula of the type (6.14).

All our generalizations of the Fisher-Hartwig formula involve a term of the form $e^{Q^2 \log n}$ as the first correction to the leading order behaviour $e^{c_0 n}$. However again when written as an average of the type (6.10) the exponential of a quadratic in k again results. Consider for example (3.13). With $\{c_n\}$ specified by (6.12) we have

$$\left\langle e^{ika(\theta)+ik\beta \sum_{j=1}^{R} q_j \log|e^{i\theta}-e^{i\phi_j}|} \right\rangle_{\mathrm{C}\beta\mathrm{E}_n} \underset{n\to\infty}{\sim} e^{ikc_0 n} e^{-k^2(\beta/2)(\sum_{j=1}^{R} q_j^2)\log n}$$

and thus, as first noted in [37], with

$$A = \sum_{l=1}^{N} \Big(a(\theta_l) + \beta \sum_{j=1}^{R} q_j \log|e^{i\theta_l} - e^{i\phi_j}| \Big)$$

the asymptotic form of the corresponding distribution is given by the Gaussian (6.13) with

$$\mu = nc_0, \qquad \sigma^2 = \beta\Big(\sum_{j=1}^{R} q_j^2\Big) \log n.$$

Thus the variance diverges logarithmically. This class of Gaussian fluctuation theorem has found use in the application of random matrix theory to the study of the statistical properties of the zeros of the Riemann zeta function [38, 39]. The study of the statistical properties of the zeros of families of L-functions requires averages over the different classical groups [40, 41, 42, 43]. We might anticipate that our new results of Section 4 will find application in this topic.

Of course it remains to prove the conjectures of this paper. Of these, Conjecture 2 is the most general, as it involves Fisher-Hartwig type parameters $\{q_j\}$, $\{b_j\}$ as well as the log-gas type parameter β. It is also of interest to extend Conjectures 3, 8 and 9 to this level of generality. Another direction of generality is to extend the domain of integration from a circle or line to a two-dimensional region [44, 45].

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